Estimation and Prediction for the Inverse Weibull Distribution under an Adaptive Type-II Progressivelly Censored Data

Ahmed A. Soliman, Essam A. Ahmed, Nasser A. Abou-Elheggag and Samah M. Ahmed

Mathematics Department, Sohag University, Sohag 82524, Egypt

Abstract: In this article, we consider the problem of estimating some lifetime parameters based on adaptive progressive Type-II censored sample from the inverse Weibull distribution. Maximum likelihood (ML) and Bayesian approaches are used to estimate the unknown parameters, coefficient of variation, reliability and hazard functions. The Bayes estimators are obtained using both symmetric and asymmetric loss functions. However, the Bayes estimators do not exist in an explicit form, Markov Chain Monte Carlo (MCMC) method is used to generate samples from the posterior distribution. Gibbs sampling within Metropolis. Hastings is applied to estimate the lifetime parameters. Furthermore, asymptotic normality of the ML and MCMC method are employed to construct the corresponding confidence intervals. The delta method is used to estimate the variances of coefficient of variation, reliability and hazard functions. Further, Bayesian two-sample prediction of the future order statistics as well as the future lower record values are discussed. Proposed methods of estimation and prediction are compared using Monte Carlo simulation study. Finally a real data set is analyzed for illustration purposes.

Keywords:Inverse Weibull distribution; Adaptive Type-II progressive censoring; Maximum likelihood estimation; Bayesian estimation; Symmetric and asymmetric loss functions; Markov Chain Monte Carlo.

1 Introduction

The inverse Weibull (IW) distribution is received some attention in the literature. Keller and Kamath [1] study the shapes of the density and failure rate functions for the basic inverse model. If the random variable Y has a Weibull distribution, then the random variable $X = Y^{-1}$ has an IW distribution with probability density function (pdf) given by

$$f(x;\beta,\lambda) = \lambda\beta x^{-(\beta+1)}e^{-\lambda x^{-\beta}} \quad ,\beta > 0, \lambda > 0, x > 0. \tag{1}$$

Notice that here β is a shape parameter which governs the shape of the distribution and λ is a scale parameter which governs the dispersion of the distribution. The corresponding cumulative distribution function (cdf) is given by

$$F(x;\beta,\lambda) = e^{-\lambda x^{-\beta}}. \tag{2}$$

The reliability and failure rate functions of IW distribution, respectively, are given by

$$S(t;\beta,\lambda) = 1 - e^{-\lambda t^{-\beta}}, \beta > 0, \lambda > 0, t > 0, \tag{3}$$

and

$$H(t;\beta,\lambda) = \lambda\beta t^{-\beta-1}e^{-\lambda t^{-\beta}}\left(1 - e^{-\lambda t^{-\beta}}\right)^{-1}. \tag{4}$$

It is observed that the hazard function of IW distribution can be monotonic decreasing or unimodal depending upon the values of β. When $\beta = 1$ and $\beta = 2$, the IW pdfs are referred to as the inverse exponential and inverse Raleigh pdfs respectively. For some more discussions on various properties of this distribution (see Kundu and Howlader [2]).

IW distribution plays an important role in many applications, including the dynamic components of diesel engines and several data sets such as the times to breakdown of an insulating fluid subject to the action of a constant tension (see Jiang et al. [3], and Nelson [4] for more practical applications). Extensive work is done on the IW distribution, for example, Calabria and Pulcini [5] give an elucidation of the IW distribution in the context of the loadstrength relationship for a component. Maswadah [6] is fitted IW distribution to the flood data reported in Dumonceaux and Antle [7].

The coefficient of variation (CV) is used in numerous areas of science such as biology, economics, and psychology, and in engineering in queueing and reliability theory (see, for example, Nairy and Rao [8] gave a summary of uses of the CV in a number of areas. Given a set of observations from IW(β, λ), the sample CV is often estimated by the ratio of the sample standard deviation to the sample mean. Or equivalent,

$$CV(\beta) = \frac{\left[\Gamma(1 - \frac{2}{\beta}) - \Gamma^2(1 - \frac{1}{\beta})\right]^{1/2}}{\Gamma(1 - \frac{2}{\beta})}, \ \beta > 2. \tag{5}$$

The theory of reliability and life testing experiments is generally applied to analyze various data arising from diverse fields of studies such as agricultural, medicine, economics, industrial and survival analysis. Many of such data are observed using some censoring methodologies. In this connection Type-I and Type-II censoring are treated as the most common and primary censoring schemes in literature. One of the drawbacks of these schemes is that they do not allow removing the units from the experiment at any time point other than the terminal point. To deal with this problem, a more general censoring scheme called progressive Type-II censoring is used. Progressive Type-II censoring scheme can be described as follows: consider an experiment in which n units are placed on a life testing experiment and m is a predetermined number of units to be failed. At the time of the first failure $x_{1:m:n}$, R_1 units are randomly removed from the remaining $n-1$ surviving units. Similarly, at the time of the second failure $x_{2:m:n}$, R_2 units of the remaining $n-2-R_1$ units are randomly removed and so on. At the time of the mth failure $x_{m:m:n}$, all the remaining $n-m-R_1-R_2-...-R_{m-1}$ units are removed. The progressively censoring scheme $R_1, R_2, ..., R_m$ are fixed and predetermined prior to the study. Several generalizations such as hybrid, progressive Type-I, progressive first failure, adaptive progressive Type-II, etc. are also extensively studied by different researchers using various lifetime distributions. One may refer to Balakrishnan and Aggarwala [9] and Balakrishnan and Kundu [10] for a detailed review of work done, particularly, on progressive and hybrid censoring.

Many articles are considered IW distribution under different censoring schemes. Among others, Kundu and Howalder [11] considered the Bayesian inference and prediction of the IW distribution for type-II censored data, Calabria and Pulcini [12] are discussed the MLE and least squares estimations of its parameters. Singh [13] are discussed the classical as well as Bayesian estimation procedures for the estimation of the unknown parameters of the IW distribution under conventional Type-I and Type-II censoring schemes. Xiuyun and Zaizai [14] established the Bayesian estimation and prediction for the IW distribution under general progressive censoring.

For the purpose of increasing the efficiency of statistical analysis as well as saving the total test time, Ng et al. [15] introduced an adjustment of progressive Type-II hybrid censoring scheme, so called adaptive progressive Type-II censoring scheme, and analyzed the data under the assumptions of exponential lifetime distribution of the experimental units. Under this scheme, the number of observed failures m is fixed in advance but the experimental time is allowed to run over the (pre-fixed) threshold time $T > 0$. If $X_{m:m:n} < T$, the experiment stops at time $X_{m:m:n}$, and we will have a usual progressive Type-II censoring scheme with the pre-fixed progressive censoring scheme $(R_1, R_2, ..., R_m)$. If $X_{J:m:n} < T < X_{J+1:m:n}$, where $J+1 < m$, we adapt the number of items progressively removed from the experiment upon failure by setting $R_{J+1} = R_{J+2} = ... =$

$R_{m-1} = 0$ and $R_m = n-m-\sum_{i-1}^{J} R_i$.Thus, the effectively applied scheme is $(R_1, R_2, ..., R_J, 0, ..., 0, n-m-\sum_{i-1}^{J} R_i)$, where $J = max\{j : X_{j:m:n} < T\}$, that is, the first observed failure time exceeding the ideal total time T. Put another way, as long as the failures occur before time T, the initially planned progressive scheme is applied. After passing time T, we do not withdraw any items at all except for the time of the mth failure where all remaining surviving items are removed. This determination results in terminating the experiment as soon as the $(J+1)$th failure time is greater than T, and the total test time will not be too far away from time T. If $T = 0$, the scheme will lead us to the case of the conventional Type-II censoring scheme, and if $T \to \infty$, we will have a usual progressive Type-II censoring scheme. This approach illustrates how an experimenter can control the experiment. The experimenter can decide to change the value of T as a compromise between a shorter experimental time and a higher chance to observe extreme failures.

Based on the adaptive progressively censored data, several papers are appeared to estimate the unknown parameter for different distributions. For example, Lin et al. [16], discussed the ML and approximate MLEs for the Weibull distribution. Hemmati and Khorram [17], studied the ML and approximate MLEs for the log-normal distribution. Mahmoud et al. [18], investigated the estimations and Bayes estimates of the unknown parameters of Pareto distribution. Ashour and Nassar [19] showed the MLE and asymptotic confidence intervals in the presence of competing risks. Ismail [20], considered estimation problems of Weibull distribution under step-stress partially accelerated life test model. AL Sobhi and Soliman [21], discussed the problem of estimating parameters of the exponentiated Weibull distribution. Recently, Nassar and Kasem [22], discussed the estimation problem of the unknown parameters of the IW distribution based on adaptive Type-II progressively hybrid censored data. They used classical and Bayesian estimation methods to estimate the unknown parameters. They obtained the Bayes estimates based on squared error loss function under the assumption of independent gamma priors using Lindley's approximation.

In this work, we propose and evaluate the performance of different estimators for the unknown parameters as well as some life parameters (reliability function, hazard function and the CV) of the IW model, under adaptive Type-II progressively censored data. Many fields of practical studies including life testing experiments require different methods to develop prediction inference in different sampling framework. Based on the observed adaptive progressively censored data, the Bayesian two sample prediction interval of a future order statistics as well as lower record values is considered as a second goal of this paper.

The layout of this paper is organized as follows: In Section 2, we obtain the MLE of β, λ, CV, $S(t)$ and $H(t)$. We also construct asymptotic intervals of unknown parameters based on the observed Fisher information matrix. In Section 3 Bayes estimates are obtained under the squared error, LINEX and general entropy loss functions. In Section 4, the problem of prediction is considered under Bayesian framework. To compare the MLE and the Bayes estimator of the all unknown parameters, Monte Carlo simulation study is performed in Section 5. Dumonceaux and Antle [7] introduced a data set on the maximum flood levels data in millions of cubic feet per second for the Susquehanna River at Harrisburg, Pennsylvania, over 20 four-year periods from 1890–1969. The analysis for the real data set is included in Section 6. Finally, in Section 7, we conclude the paper.

2 Likelihood Inference and Information Matrix

This section discusses the process of obtaining the MLE of the unknown parameters β and λ as well as some lifetime parameters CV, $S(t)$ and $H(t)$ based on adaptive progressive type-II censoring censored data. Both point and interval estimations of the parameters are derived.

Given censoring scheme $R = (R_1, R_2, ..., R_J, 0, 0, ..., 0, n-m-\sum_{i-1}^{J} R_i)$, where $J = max\{j : X_{j:m:n} < T\}$, the likelihood function of the observed data $x_1, x_2, ..., x_m$ is expressed by

$$\begin{aligned} L(\underline{x};\beta,\lambda|J &= j) = c(R)\prod_{i=1}^{m} f(x_i;\beta,\lambda)\prod_{i=1}^{j}\left[1-F\left(x_i;\beta,\lambda\right)\right]^{R_i}\left[1-F(x_m;\beta,\lambda))\right]^{R_m^*} \\ &= c(R)\prod_{i=1}^{m}\lambda\beta x_i^{-\beta-1}\exp(-\lambda x_i^{-\beta})\prod_{i=1}^{j}\left(1-\exp(-\lambda x_i^{-\beta})\right)^{R_i}\left[1-\exp(-\lambda x_m^{-\beta})\right]^{R^*}, \end{aligned} \tag{6}$$

where $R^* = n - m - \sum_{i-1}^{j} R_i$ and

$$c(R) = n(n-R_1-1)(n-R_1-R_2-2)...(n-R_1-R_2-...R_j-1)(n-R_1-R_2-...R_j-2)...(n-R_1-R_2-...R_j-m+1),$$

is the normalizing constant and .

Taking the logarithm of $L(\underline{x};\beta,\lambda|J=j)$ and ignoring the additive constant, we obtain

$$\ell = m\log\beta\lambda - (\beta+1)\sum_{i=1}^{m}\log x_i - \lambda\sum_{i=1}^{m}x_i^{-\beta} + \sum_{i=1}^{j}R_i\log\left(1-\exp(-\lambda x_i^{-\beta})\right) + R^*\log\left(1-\exp(-\lambda x_m^{-\beta})\right). \tag{7}$$

By differentiating the associated ℓ with respect to β and λ and equating them to zero, we obtain the following likelihood equations

$$\frac{m}{\beta} - \sum_{i=1}^{m}\log x_i + \lambda\sum_{i=1}^{m}x_i^{-\beta}\log x_i - \lambda\sum_{i=1}^{j}R_i x_i^{-\beta}w_i\log x_i - \lambda R^* x_m^{-\beta}w_m\log x_m = 0, \tag{8}$$

and

$$\frac{m}{\lambda} - \sum_{i=1}^{m}x_i^{-\beta} + \sum_{i=1}^{j}R_i x_i^{-\beta}w_i + R^* x_m^{-\beta}w_m = 0, \tag{9}$$

where $w_i = \left[\exp(\lambda x_i^{-\beta}) - 1\right]^{-1}$.

We observed that these expressions are not in closed form. Therefore, MLEs can be secured through iterative procedure. Here, we suggest to use Newton Raphson method.

If $\hat{\lambda}_{ML}$ and $\hat{\beta}_{ML}$ be the MLE's of unknown parameters β and λ respectively. Therefore, the MLE of the CV, the reliability function and hazard rate function for a specified time t can be expressed as;

$$\widehat{CV}_{ML} = \frac{\left[\Gamma(1-\frac{2}{\hat{\beta}_{ML}}) - \Gamma^2(1-\frac{1}{\hat{\beta}_{ML}})\right]^{1/2}}{\Gamma(1-\frac{2}{\hat{\beta}_{ML}})}, \tag{10}$$

$$\hat{S}_{ML}(t) = 1 - e^{-\hat{\lambda}_{ML}t^{-\hat{\beta}_{ML}}}, \tag{11}$$

and

$$\hat{H}_{ML}(t) = \hat{\lambda}_{ML}\hat{\beta}_{ML}t^{-(\hat{\beta}_{ML}+1)}e^{-\hat{\lambda}_{ML}t^{-\hat{\beta}_{ML}}}\left(1-e^{-\hat{\lambda}_{ML}t^{-\hat{\beta}_{ML}}}\right)^{-1}. \tag{12}$$

Also, In this section, we obtained the Fisher information matrix for constructing 95% asymptotic confidence interval for the parameters. The Fisher information matrix can be obtained by

$$I^{-1} = \begin{bmatrix} var(\hat{\beta}) & cov(\hat{\lambda},\hat{\beta}) \\ cov(\hat{\beta},\hat{\lambda}) & var(\hat{\lambda}) \end{bmatrix} = \begin{bmatrix} -E\left(\frac{\partial^2\ell}{\partial\beta^2}\right) & -E\left(\frac{\partial^2\ell}{\partial\beta\partial\lambda}\right) \\ -E\left(\frac{\partial^2\ell}{\partial\lambda\partial\beta}\right) & -E\left(\frac{\partial^2\ell}{\partial\lambda^2}\right) \end{bmatrix}^{-1}_{(\beta=\hat{\beta}_{ML},\lambda=\hat{\lambda}_{ML})}, \tag{13}$$

where

$$\begin{aligned} \frac{\partial^2\ell}{\partial\beta^2} &= -\frac{m}{\beta^2} - \lambda\sum_{i=1}^{m}x_i^{-\beta}\log x_i + \lambda\sum_{i=1}^{j}R_i x_i^{-\beta}w_i\log^2 x_i\left[1-\lambda x_i^{-\beta}\exp(\lambda x_i^{-\beta})w_i\right] \\ &\quad +\lambda R^* x_m^{-\beta}w_m\log^2 x_m\left[1-\lambda x_m^{-\beta}\exp(\lambda x_m^{-\beta})w_m\right], \end{aligned} \tag{14}$$

$$\frac{\partial^2 L(\lambda,\beta)}{\partial\lambda^2} = \frac{-m}{\lambda^2} - \sum_{i=1}^{j} R_i x_i^{-2\beta}\exp(\lambda x_i^{-\beta})w_i^2 - R^* x_m^{-2\beta}\exp(\lambda x_i^{-\beta})w_m^2, \tag{15}$$

$$\frac{\partial^2\ell}{\partial\beta\partial\lambda} = \frac{\partial^2\ell}{\partial\lambda\partial\beta} = \sum_{i=1}^{m} x_i^{-\beta}\log x_i - \sum_{i=1}^{j} R_i x_i^{-\beta} w_i(1-x_i^{-\beta}w_i)\log x_i - R^* x_m^{-\beta} w_m(1-x_m^{-\beta}w_m)\log x_m, \tag{16}$$

Unfortunately, the exact mathematical expressions for the above expectations are very difficult to obtain. Therefore, the approximate confidence intervals of the parameters are derived based on the asymptotic distributions of the MLEs of the elements of the vector of unknown parameters β and λ. It is known that the asymptotic distribution of the MLEs of β and λ is given by Miller [23].

$$\left(\left(\hat{\beta}-\beta\right), \left(\hat{\lambda}-\lambda\right)\right) \to N\left(0, I^{-1}\right),$$

where $I^{-1}(\hat{\beta},\hat{\lambda})$ is the variance-covariance matrix of the unknown parameters β and λ, obtained by (13) dropping the expectation operator E. For large value of effective sample size m, the approximate $100(1-\gamma)\%$ two sided confidence intervals for β and λ are respectively given by

$$\hat{\beta} \pm Z_{1-\frac{\gamma}{2}}\sqrt{\text{var}(\hat{\beta})} \text{ and } \hat{\lambda} \pm Z_{1-\frac{\gamma}{2}}\sqrt{\text{var}(\hat{\lambda})}, \tag{17}$$

where Z_q is the $100q-th$ percentile of a standard normal distribution.

In order to find the approximate estimator of the variance of $\widehat{CV}_{ML}$, $\hat{S}_{ML}(t)$ and $\hat{H}_{ML}(t)$, we use the delta method. The delta method is a general approach for computing confidence intervals for functions of MLEs, see (Greene [24]; Agresti [25]). Let

$$G_1 = \left(\frac{\partial CV}{\partial\beta}, \frac{\partial CV}{\partial\lambda}\right), G_2 = \left(\frac{\partial S(t)}{\partial\beta}, \frac{\partial S(t)}{\partial\lambda}\right), G_3 = \left(\frac{\partial H(t)}{\partial\beta}, \frac{\partial H(t)}{\partial\lambda}\right). \tag{18}$$

Then the approximate estimator of $var\left(\hat{S}_{ML}(t)\right)$, $var\left(\hat{H}_{ML}(t)\right)$ and $var(\widehat{CV}_{ML})$ are given, respectively, by

$$v\hat{a}r\left(\widehat{CV}\right) \simeq \left[G_1 I^{-1} G_1^T\right]\big|_{(\hat{\beta},\hat{\lambda})}, v\hat{a}r\left(\hat{S}(t)\right) \simeq \left[G_2 I^{-1} G_2^T\right]\big|_{(\hat{\beta},\hat{\lambda})} \text{ and } v\hat{a}r\left(\hat{H}(t)\right) \simeq \left[G_3 I^{-1} G_3^T\right]\big|_{(\hat{\beta},\hat{\lambda})} \tag{19}$$

Where, G_i^T is the transpose of G_i , $i = 1, 2, 3$. Theses results yields the approximate confidence intervals for $S(t)$, $H(t)$ and CV as

$$\widehat{CV}_{ML} \pm z_{\gamma}\sqrt{var(\widehat{CV})}, \hat{S}(t) \pm z_{\gamma}\sqrt{var(\hat{S}(t))} \text{ and } \hat{H}(t) \pm z_{\gamma}\sqrt{var(\hat{H}(t))}. \tag{20}$$

3 Bayes Estimation

In this section we describe how to obtain the Bayes estimates and the corresponding credible intervals of the unknown parameters and any function of the them (coefficient of variation CV, reliability $S(t)$ and hazard function $H(t)$).

In Bayesian analysis the parameter of interest is to be considered as a random variable and follows some prior distribution. Here both the parameters λ, β are unknown and it is difficult to obtain joint bivariate prior distribution of the them. So we assume that λ and β are independent and a priori distributed as gamma G(a, b) and G(c, d) distributions respectively with a, b, c and d denoting hyperparameters, which assumed to be known. The above considered prior may be regarded as a non-informative prior by setting the values of hyper-parameters is to be zero. Therefore the considered joint prior for (λ, β) is then obtained as

$$\pi(\lambda,\beta) = \lambda^{a-1}\beta^{c-1}e^{-(b\lambda+d\beta)}; a, b, c \text{ and } d \geq 0. \tag{21}$$

Then by using equation (6) and (21) the joint posterior after simplification can be given as

$$p(\lambda,\beta|\underline{x}) \propto \lambda^{m+a-1}\beta^{m+c-1}e^{-(b\lambda+d\beta)}e^{-\beta\sum\limits_{i=1}^{m}\log x_i}e^{-\lambda\sum\limits_{i=1}^{m}x_i^{-\beta}}\prod_{i=1}^{j}\left(1-\exp(-\lambda x_i^{-\beta})\right)^{R_i}\left[1-\exp(-\lambda x_m^{-\beta})\right]^{R^*}. \tag{22}$$

We observe that $p(\lambda,\beta|\underline{x})$ is analytically intractable and moreover the Bayes estimator of some parametric function of $(\lambda,\,\beta)$ involves ratio of two integrals. Thus when the Bayes estimates are obtained some approximation methods should be employed in order to solve the corresponding ratio of integrals. In such a situation, the most appropriate MCMC methods namely Gibbs sampler and Metropolis–Hastings (MH) Algorithm can be effectively used. The MH algorithm generate samples from an arbitrary proposal distribution. We assume that proposal distributions for and are independent normal and then compute the desired Bayes estimates. Samples generated from the posterior distribution are further used in the construction of highest posterior density intervals for unknown parameters. For implementing the Gibbs with in MH algorithm, the full conditional posterior densities of λ and β are given by

$$p_1(\lambda|\underline{x},\beta) \propto \lambda^{m+a-1}e^{-\lambda\left(b+\sum\limits_{i=1}^{m}x_i^{-\beta}\right)}\prod_{i=1}^{j}\left(1-\exp(-\lambda x_i^{-\beta})\right)^{R_i}\left[1-\exp(-\lambda x_m^{-\beta})\right]^{R^*}. \tag{23}$$

and

$$p_2(\beta|\underline{x},\lambda) \propto \beta^{m+c-1}e^{-\beta\left(d+\sum\limits_{i=1}^{m}\log x_i\right)}e^{-\lambda\sum\limits_{i=1}^{m}x_i^{-\beta}}\prod_{i=1}^{j}\left(1-\exp(-\lambda x_i^{-\beta})\right)^{R_i}\left[1-\exp(-\lambda x_m^{-\beta})\right]^{R^*}, \tag{24}$$

and the following steps are required to generate samples from the given posterior distribution:

Step 1: Choose an initial guess of $(\lambda,\,\beta)$, say $(\lambda^{(0)}=\hat{\lambda}_{ML},\,\beta^{(0)}=\hat{\beta}_{ML})$.

Step 2: Set $i=1$.

Step 3: Generate $\lambda^{(i)}$ from (27) using Metropolis-Hastings algorithm with proposal distribution $q(\lambda)= N(\lambda^{(i-1)},var(\hat{\lambda}))$ as follows:

a) Let $\varepsilon=\lambda^{(i-1)}$

b) Generate λ^* from the proposal distribution.

c) Accept λ^* with probability $\rho\left(\varepsilon,\lambda^*\right)=\min[1,\frac{q(\varepsilon)p_1(\lambda^*|\underline{x},\beta)}{q(\lambda^*)p_1(\varepsilon|\underline{x},\beta)}]$, or accept ε with 1-$\rho\left(\varepsilon,\lambda^*\right)$.

Step 4: Generate $\beta^{(i)}$ from (28) using MH algorithm with proposal distribution $q(\beta)= N(\beta^{(i-1)},var(\hat{\beta}))$

Step 5: From Equations (3)-(5), compute $CV\left(\beta^{(i)}\right)$, $S\left(t;\beta^{(i)},\lambda^{(i)}\right)$ and $H\left(t;\beta^{(i)},\lambda^{(i)}\right)$.

Step 6: Set $i=i+1$.

Step 7: Repeat steps $3-5$ N times.

We discard the initial M number of burn-in samples and obtain estimates using the remaining $N-M$ samples. Thus Bayes estimates of λ and β under the SE loss function can be computed as follows:

$$\tilde{\theta}_{SE}=\frac{1}{N-M}\sum_{i=M+1}^{N}\theta^{(i)}.$$

Proceeding similarly we can obtain the desired estimates under the LINEX and GE loss functions, respectively, in the following form

$$\tilde{\theta}_{LINEX} = -\frac{1}{c}\log\left[\frac{1}{N-M}\sum_{i=M+1}^{N} e^{-c\theta^{(i)}}\right], \tag{25}$$

and

$$\tilde{\theta}_{GE} = \left[\frac{1}{N-M}\sum_{i=M+1}^{N}\theta^{-\rho}\right]^{-1/\rho}. \tag{26}$$

where M is burn-in and $\theta = \lambda$, β, $CV(\beta)$, $S(t;\beta,\lambda)$ or $H(t;\beta,\lambda)$.

Step 8: To obtain Highest Posterior Density (HPD) interval of λ , we order $\{\lambda^{(i)}\}$ as $\lambda_{(1)} < ... < \lambda_{(N)}$. Chen and Shao [26] provided a simple method for constructing a $100(1-\gamma)\%$ HPD credible interval based on MCMC samples. Let $\lambda_{(i)}$ be the ith smallest of λ_i and denote $I_i = (\lambda_{(i)},\ \lambda_{(i+[(N-M)\times(1-\gamma)])})$ for $i = M+1, ..., (N-M) - [(N-M)\times(1-\gamma)]$. Then I_i with the smallest width among all $I_i's$ is chosen as the $100(1-\gamma)\%$ HPD credible interval for λ. Similarly, we can obtain the HPD credible interval for β, $CV(\beta)$, $S(t;\beta,\lambda)$ and $H(t;\beta,\lambda)$.

In the next section Bayesian prediction is discussed

4 Bayesian Two-sample prediction

In many business and engineering applications, the experimenters usually wish to predict the future observations in a population, based on existing data. Here we discuss Bayesian prediction in two-sample situations. We present the Bayesian predictive distribution for the future order statistics as well as a future record values based on the observed adaptive Type-II progressive-censored data. It is assumed that only the first m adaptive Type-II progressive-censored observations $\mathbf{x} = (x_1, x_2, ..., x_m)$ are observed and we wish to predict the future sample (order statistics or lower record values) $\mathbf{y} = (y_1, y_2, ..., y_k)$ of size k from the same population.

4.1 Bayesian Prediction Interval for Future Order Statistics

Let $\mathbf{x} = (x_1, x_2, ..., x_m)$ be a given sample of an adaptive Type-II progressive censored order statistics from IW (λ, β) distribution and $(y_1 < y_2 < ... < y_k)$ be a future order sample of size k taken also from the same distribution. We aim to derive prediction interval of $y = \{y_1 < y_2 < ... < y_k,\}$.

The marginal density function of the sth order statistics $y_s, s = 1, 2, ..., k$ is given by

$$\begin{aligned}
g(y_s|\lambda,\beta) &= \frac{k!}{(k-s)!(s-1)!}\left[F(y_s|\lambda,\beta)\right]^{s-1}\left[1-F(y_s|\lambda,\beta)\right]^{(k-s)} f(y_s|\lambda,\beta) \\
&= \frac{k!}{(k-s)!(s-1)!}\sum_{j=0}^{k-s}\binom{k-s}{j}(-1)^j\left[F(y_s|\lambda,\beta)\right]^{s+j-1} f(y_s|\lambda,\beta) \\
&= \frac{k!}{(k-s)!(s-1)!}\lambda\beta y_s^{-\beta-1}\sum_{j=0}^{k-s}\binom{k-s}{j}(-1)^j e^{-(s+j)\lambda y_s^{-\beta}}.
\end{aligned} \tag{27}$$

Then, the Bayesian predictive density function of y_s given x is obtained as follows

$$\hat{g}(y_s|x) = \int_0^\infty\int_0^\infty g(y_s|\lambda,\beta)p(\lambda,\beta|\underline{x})d\lambda d\beta, \tag{28}$$

where $p(\lambda,\beta|\underline{x})$ is the joint posterior density of λ and β as given in (22). It is immediate that $\hat{g}(y_s|x)$ can not be expressed in closed form and hence it can not be evaluated analytically. Based on the Monte Carlo method, (28) can be approximated by

$$\hat{g}(y_s|x) \sim \frac{1}{N}\sum_{i=1}^{N} g(y_s|\lambda_i,\beta_i), \tag{28}$$

where (λ_i, β_i), $i = 1, \ldots, N$ are generated from from $p(\lambda, \beta|\underline{x})$ by the Gibbs sampling strategy as described in Section (3). Then the predictive reliability function is established by

$$\begin{aligned} G(y_s|x) &\approx \frac{1}{N}\sum_{i=1}^{N}\int_{y_s}^{\infty} g(z|\lambda_i, \beta_i)dz \\ &= \frac{k!}{N(k-s)!(s-1)!}\sum_{i=1}^{N}\sum_{j=0}^{k-s}\binom{k-s}{j}(-1)^j\left[\frac{1-e^{-\lambda(j+s)y_s^{-\beta}}}{(j+s)}\right]. \end{aligned} \tag{29}$$

Then, two sided symmetric $100(1-\gamma)\%$ Bayesian prediction bounds for y_s are obtained by solving the following equations with respect to y_s:

$$\frac{k!}{N(k-s)!(s-1)!}\sum_{i=1}^{N}\sum_{j=0}^{k-s}\binom{k-s}{j}(-1)^j\left[\frac{1-e^{-\lambda(j+s)y_s^{-\beta}}}{(j+s)}\right] = \frac{\gamma}{2} \tag{30}$$

and

$$\frac{k!}{N(k-s)!(s-1)!}\sum_{i=1}^{N}\sum_{j=0}^{k-s}\binom{k-s}{j}(-1)^j\left[\frac{1-e^{-\lambda(j+s)y_s^{-\beta}}}{(j+s)}\right] = 1-\frac{\gamma}{2} \tag{31}$$

4.2 Bayesian Prediction Interval for Future Record Values

Record values and associated statistics are of great important in several real live problems involving weather, economic, and sport data. Many properties and applications of record values have appeared in the statistical literature, among them, see Arnold et al. [27] and Gulati and Padgett [28]. Furthermore, many daily life fields such as clinical trials, insurance, industry marketing and others, indicate that Bayesian prediction of record values has an extraordinary enthusiasm, as the best approach found as of not long ago. Hence, Bayesian prediction bounds for some record statistics based on different distributions have been studied by a number of statisticians, among them, see Ali Mousa et al. [29], Ahmadi and Doostparast [30], Kizilaslan and Nadar [31], Dey et al. [32] Singh et al. [33] and Shafay et al. [34].

An observation x_i will be called a lower record values if its value is less than all previous observations. Thus x_i is a lower record values if $x_i < x_j$ for every $i > j$. In this subsection, we aim to predict the Bayesian prediction intervals of the future lower record values based on the observed adaptive Type-II progressive-censored values. We assume that $(x_1, x_2, ..., x_m)$ are the observed adaptive progressively type-II censored order statistics from IW(λ, β) distribution, and $(z_{L(1)}, z_{L(2)}, ..., z_{L(r)})$ are the first r lower records from a future sequence from the same distribution. Suppose that we are interested in the predictive density of the lower record $z_{L(s)}$, $1 \leq s \leq r$.

The probability density function of the s^{th} lower record is given (see Ahsanullah, [35]) by

$$\begin{aligned} h(z_s|\lambda, \beta) &= \frac{1}{(s-1)!}\{-\log F(z_s|\lambda, \beta)\}^{s-1} f(z_s|\lambda, \beta) \\ &= \frac{1}{(s-1)!}\lambda^s \beta z_s^{-s\beta-1} e^{-\lambda z_s^{-\beta}}, \end{aligned} \tag{32}$$

and the Bayesian predictive density function of $z_{L(s)}$ given x is obtained as follows

$$h^*(z_s|x) = \int_0^{\infty}\int_0^{\infty} h(z_s|\lambda, \beta)p(\lambda, \beta|\underline{x})d\lambda d\beta, \tag{33}$$

As before, based on the MCMC samples $\{(\lambda_i, \beta_i), i = 1, 2, ..., N\}$ (33) can be approximated by

$$h^*(z_s|x) \sim \frac{1}{N}\sum_{i=1}^{N} h(z_s|\lambda_i, \beta_i). \tag{34}$$

Hence, the predictive reliability function is established by

$$\begin{aligned} H(z_s|x) &\approx \frac{1}{N}\sum_{i=1}^{N}\int_{z_s}^{\infty} h(t|\lambda_i,\beta_i)dt \\ &= \frac{1}{N\,(s-1)!}\sum_{i=1}^{N}\int_{z_s}^{\infty} \lambda^s \beta z_s^{-s\beta-1} e^{-\lambda z_s^{-\beta}} dz_s \end{aligned} \tag{35}$$

Moreover, the two sided symmetric $100(1-\gamma)\%$ Bayesian prediction bounds for y_s are obtained by sby solving following equations simultaneously:

$$\frac{1}{N\,(s-1)!}\sum_{i=1}^{N}\int_{z_s}^{\infty} \lambda^s \beta z_s^{-s\beta-1} e^{-\lambda z_s^{-\beta}} dz_s = \frac{\gamma}{2} \tag{36}$$

and

$$\frac{1}{N\,(s-1)!}\sum_{i=1}^{N}\int_{z_s}^{\infty} \lambda^s \beta z_s^{-s\beta-1} e^{-\lambda z_s^{-\beta}} dz_s = 1-\frac{\gamma}{2} \tag{37}$$

In the next section a Monte Carlo simulation study is performed to compare the proposed estimation methods.

5 Simulation Results

In this section, we conduct a Monte Carlo simulation study to compare the performance of proposed method of estimation and prediction. For computation purposes unknown parameters are assigned as $\beta = 3$ and $\lambda = 3$. Then an adaptive Type-II progressive-censored sample are generated using various censoring schemes $(n, m, R; T)$. mean SEs (MSEs) of all estimators for a given adaptive Type-II progressive-censored sample are computed based on 1000 replications of it. We compute MLEs of unknowns using the proposed Newtion Rafson algorithm. Bayes estimates are obtained using MH algorithm. Through, MH algorithm, we take into account the MLEs as initial guess values, and the associated variance–covariance matrix of β , λ. All Bayes estimates are obtained with respect to gamma prior distributions under the SE, LINEX and GE loss functions. We take hyperparameters as $a = 3$; $b = 1$; $c = 3$ and $d = 1$ to compute various Bayes estimates. In Tables $5-8$, we report estimated values of both unknown parameters along with associated MSEs for different censoring schemes. In these tables for each estimation methods the first value denotes the average estimate of the respective unknown parameter and the immediate lower value denotes the MSE of corresponding estimator. Different combinations of $(n; m; T)$ are taken into consideration for computation purposes. From tabulated values it is observed that based on MSEs, higher values of n and m lead to better estimates. In Table 9 we present the 95% coverage probabilities of unknown parameters β and λ as well as CV, $S(t)$ and $H(t)$ for different adaptive Type-II progressive-censored samples. We have computed 95% asymptotic and HPD intervals of the unknown parameters β and λ as well as CV, $S(t)$ and $H(t)$. We observe that HPD intervals of and perform better than asymptotic intervals as far as average interval length is concerned. We generated 500 an adaptive type-II progressive samples of size $m = 40$ from the IW distribution with the values of parameters $(3, 3)$, $T = 0.8$, using censoring scheme $R_1 = ... = R_{10} = 1$, $R_{11} = ... = R_{40} = 0$, we constructed 95% Bayesian two-sided equitailed prediction intervals for the future order statistics and record values, from the same distribution based on different cases of censoring are presented in Tables 10 and 11.

6 Data analysis

In this section, for illustration purpose, we consider the following real data set as described in Dumonceaux and Antle (1973).

0.654, 0.613, 0.315, 0.449, 0.297, 0.402, 0.379, 0.423, 0.379, 0.324,
0.269, 0.740, 0.418, 0.412, 0.494, 0.416, 0.338, 0.392, 0.484, 0.265.

The data set represents the maximum flood levels (in millions of cubic feet per second) of the Susquehenna River at Harrisburg, Pennsylvenia over 20 four-year periods (1890 − 1969). Here, we generated the adaptive Type-II progressive censored sample from the original measurements. We took, $m = 18$, $T = 0.4$, $R_6 = 2$ and $R_i = 0$ for $i \neq 6$. Thus, the adaptive Type-II progressive censored sample based on R is

0.265, 0.269, 0.297, 0.315, 0.324, 0.338, 0.379, 0.379, 0.392,
0.402, 0.412, 0.416, 0.418, 0.449, 0.484, 0.494, 0.613, 0.654.

From the MLEs of model parameters $\beta,\ \lambda, S(t = 0.3),\ H(t = 0.3)$ and CV are obtained as

Parameter	MLE			Bayes		
	Estimate	Lower	Upper	Estimate	Lower	Upper
β	4.5864	2.9832	6.1897	4.6419	4.1691	5.1021
λ	0.0085	0	0.0247	0.0079	0.0054	0.0109
$S(t = 0.3)$	0.8820	0.7684	0.9957	0.8674	0.7160	0.9628
$H(t = 0.3)$	4.3694	1.4708	7.2680	4.5673	2.1341	7.1048
CV	0.3516	0.3499	0.3532	0.3453	0.3561	0.3605

We also compute the approximate Bayes estimates of $\beta,\ \lambda, S(t = 0.3),\ H(t = 0.3)$ and CV under both LINEX and GE loss function with $c = -1, 1$ and $q = -1, 1$ and they are in Table 1. The posterior mean, median, mode and standard deviation (SD) and skewness (Ske) of the parameters $\beta,\ \lambda, S(t = 0.3), H(t = 0.3)$and CV are obtained in Table 2.

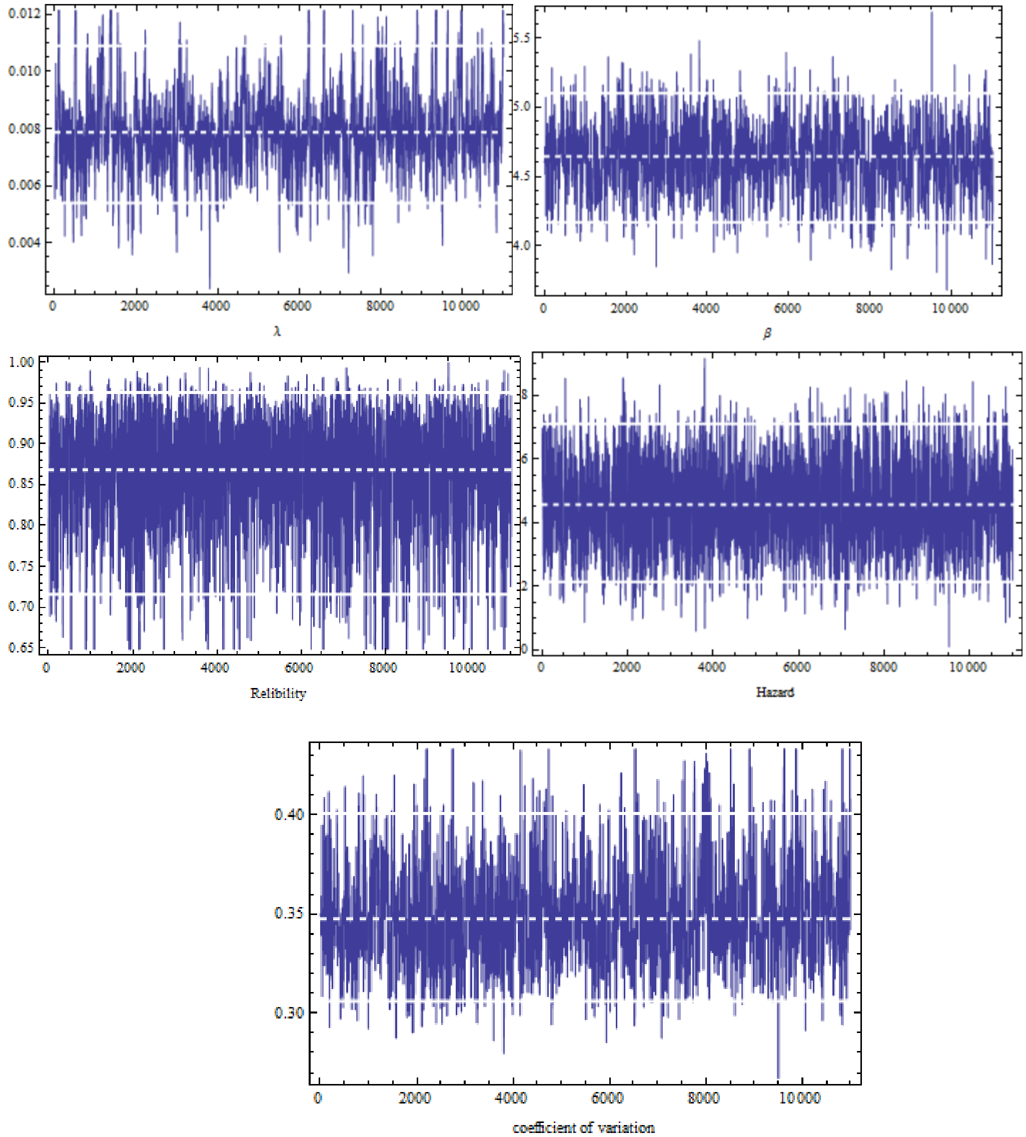

Fig.1 .Trace plots of the parameters generated by the MCMC method for real data.

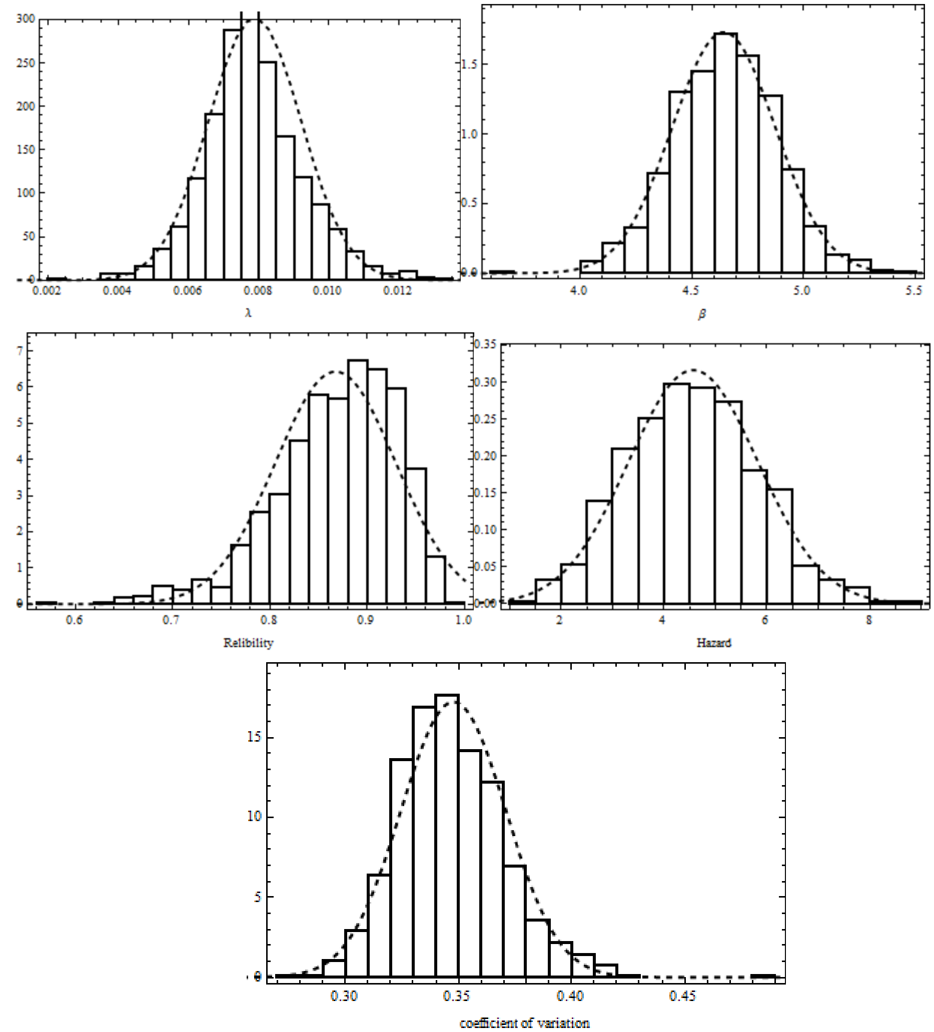

Fig 2 Histogram of the parameters generated by the MCMC method for real data.

Taple 1. Bayes MCMC estimates under LINEX and GE for real data.

parameter	LINEX		GE	
	$c=-1$	$c=1$	$q=-1$	$q=1$
λ	0.0079	0.0079	0.0079	0.0077
β	4.6697	4.6140	4.6419	4.6298
$S(t=0.3)$	0.8693	0.8654	0.8674	0.8625
$H(t=0.3)$	5.3587	3.8068	4.5674	4.1379
CV	0.3480	0.3474	0.3477	0.3461

Taple 2. MCMC results for some posterior characteristics for real data.

parameter	Mean	Mode	Median	SD	Skewness
λ	0.0079	0.0073	0.0077	0.0013	0.4139
β	4.6419	4.6478	4.6438	0.2359	-0.0305
$S(t=0.3)$	0.8674	0.8949	0.8767	0.0625	-0.8947
$H(t=0.3)$	4.5674	4.5410	4.5586	1.2600	0.0486
CV	0.3477	0.3421	0.3459	0.0239	0.4985

For this example, we generate $11,000$ MCMC samples and discard the first 1000 values as 'burn-in', based on them we compute 90% and 95% credible intervals are given in Table 3 and Table 4 for the future order statistics and future lower record values, respectively.

Table 3: Two sample prediction for the future order statistics

90% (HPD) credible intervals for Y_S			95% (HPD) credible intervals for Y_S	
Y_S	[Lower,Upper]	Length	[Lower,Upper]	Length
Y_1	$[0.2314, 0.3098]$	0.0775	$[0.2235, 0.3277]$	0.1042
Y_2	$[0.2535, 0.3402]$	0.0867	$[0.2415, 0.3490]$	0.1075
Y_3	$[0.2706, 0.3576]$	0.0870	$[0.2646, 0.3818]$	0.1172
Y_4	$[0.2818, 0.3727]$	0.0909	$[0.2652, 0.4068]$	0.1416
Y_5	$[0.2930, 0.3898]$	0.0968	$[0.2789, 0.4407]$	0.1618

Table 4: Two sample prediction for the future lower record values

90% (HPD) credible intervals for Z_S			95% (HPD) credible intervals for Z_S	
Z_S	[Lower,Upper]	Length	[Lower,Upper]	Length
Z_1	$[0.2679, 0.8342]$	0.5663	$[0.2487, 0.9946]$	0.7459
Z_2	$[0.2314, 0.4855]$	0.2541	$[0.2341, 0.4843]$	0.2502
Z_3	$[0.2264, 0.3742]$	0.1478	$[0.2194, 0.3939]$	0.1745
Z_4	$[0.2028, 0.3388]$	0.1360	$[0.2137, 0.3508]$	0.1371
Z_5	$[0.1958, 0.3090]$	0.1132	$[0.1970, 0.3211]$	0.1241

7 Concluding Remarks

In this article, we considered the ML, and the Bayesian inference and prediction for the parameters , reliability, hazard rate functions and CV of the IW distribution using the adaptive progressive Type-II censoring scheme. Also, we develop an approximate confidence intervals for the parameters , reliability, hazard rate functions and CV of the IW distribution. A simulation study is conducted to examine and compare the performance of the proposed methods. It is clear from the tables that the proposed Bayes estimates perform very well for different (n, m, and $CS\ R$). The results in the tables also reveal the superiority of the Bayesian methods as compared with the classical method when suitable prior information does become available. The results in these tables show that the MSEs decreases as the effective sample size, m, increases, as one would expect.

References

[1] A. Z. Keller and A. R. R. Kamath, Alternative reliability models for mechanical systems, Third International Conference on Reliability and Maintainability, Toulouse, France, 1982.

[2] D. Kundua and H. Howlader. Bayesian inference and prediction of the inverse Weibull distribution for type-II censored data. Computational Statistics and Data Analysis. 54, 1547-1558, (2010).

[3] R. Jiang and D. N. P. Murthy, Models involving two Inverse Weibull distributions, Reliability Engineering and System Safety. 73, 73-81 (2001).

[4] W. Nelson, Accelerated Testing: Statistical Models, Test Plans and Data Analysis. Wiley, New York, (1990).

[5] R. Calabria and G. Pulcini, Point estimation under asymmetric loss functions for left truncated exponential samples. Communications in Statistics Theory and Methods. 25, 585-600 (1996).

[6] M. Maswadah, Conditional confidence interval estimation for the inverse Weibull distribution based on censored generalized order statistics. Journal of Statistical Computation and Simulation. 73, 887-898 (3003).

[7] R. Dumonceaux and C. E. Antle, Distribution between the log-normal and the Weibull distributions, Technometrics. 15, 923-926, (1973).

[8] K. S. Nairy and K. A. Rao, Tests of coefficient of variatin of normal population Comm. Stat., Simulation and Computation. 32, 641-661 (2003).

[9] N. Balakrishnan and R. Aggarwala, Progressive censoring: theory, methods and applications, Birkhauser: Boston, M A. (2000).

[10] N. Balakrishnan and D. Kundu, Hybrid censoring: Models, inferential results and applications. Computational Statistics and Data Analysis. 57, 166-209 (2013).

[11] D. Kundua and H. Howlader, Bayesian inference and prediction of the inverse Weibull distribution for type-II censored data. Computational Statistics and Data Analysis. 54, 1547-1558 (2010).

[12] R. Calabria and G. Pulcini, On the maximum likelihood and least-squares estimation in the inverse Weibull distribution, Statistica Applicata. 2, 53-63 (1990).

[13] S. K. Singh, U. Singh and Dinesh Kumar. Bayesian estimation of parameters of inverse Weibull distribution. Journal of Applied Statistics. 40, 1597-1607 (2013).

[14] P. Xiuyu nad Y. Zaizai, Bayesian estimation and prediction for the inverse Weibull distribution under general progressive censoring, Communications in Statistics, Theory and Methods. 45, 621-635 (2016).

[15] H. K. T. Ng, D. Kundu and P. S Chan. Statistical analysis of exponential lifetimes under an adaptive Type-II progressive censoring scheme. Naval Research Logistics. 56, 687-698 (2009).

[16] C. T. Lin, H. K. T. Ng and P. S. Chan. Statistical Inference of Type-II Progressively Hybrid Censored Data with Weibull Lifetimes. Communications in Statistics-Theory and Methods. 38, 1710-1729 (2009).

[17] F. Hemmati and E. Khorram. Statistical analysis of the log-normal distribution under Type-II progressive hybrid censoring schemes. Communications in Statistics-Simulation and Computation. 42, 52-57 (2013).

[18] M. A. W. Mahmoud, A. A. Soliman, A. H. Abd Ellah, R.M. El-Sagheer, Estimation of generalized Pareto under an adaptive type-II progressive censoring, Intelligent Information Management. 5 73-83 (2013).

[19] S. K. Ashour and M. M. A. Nassar. Analysis of generalized exponential distribution under adaptive Type-II progressive hybrid censored competing risks data, International Journal of Advanced Statistics and Probability. 2, 108-113 (2014).

[20] A. A. Ismail, Inference for a step-stress partially accelerated life test model with an adaptive type II progressively hybrid censored data from Weibull distribution, Journal of Computational and Applied Mathematics. 260, 533-542 (2014).

[21] M. M. Al-Sobhi and A. A. Soliman. Estimation for the exponentiated Weibull model with adaptive type-II progressive censored schemes, Applied Mathematical Modelling. 40, 1180-1192 (2016).

[22] M. Nassar and O. E. Abo-Kasem. Estimation of the inverse Weibull parameters under adaptive type-II progressive hybrid censoring scheme, Journal of Computational and Applied Mathematics archive. 315, 228-239 (2017).

[23] G. E. Miller, Asymptotic test statistics for coefficients of variation, Commun. Statist. Theor. Meth. 20, 3351-3363 (1991).

[24] W. H. Greene, Econometric Analysis, 4th Ed. New York: Prentice Hall (2000).

[25] A. Agresti, Categorical Data Analysis. 2nd Edition. New York: Wiley (2002).

[26] M. H. Chen and Q. M. Shao, Monte Carlo estimation of Bayesian credible and HPD intervals. J Comput Graph Stat. 8, 69-92 (1999).

[27] B. C. Arnold, N. Balakrisnan and H. N. Nagaraja. Records. Wiley, (1998).

[28] S. Gulati and W. J. Padgett. Parametric and Nonparametric Inference from Record-Breaking Data (2003).

[29] M. A. M. Ali Mousa, Z. F. Jaheen and A. A. Ahmad, Bayesian estimation, prediction and characterization for the Gumbel model based on records, Statistics. 36, 65-74 (2002).

[30] J. Ahmadi and M. Doostparast, Bayesian estimation and prediction for some life time distributions based on record values, Statist. Papers. 47, 373-392 (2006).

[31] F. Kizilaslan and M. Nadar, Estimation and prediction of the Kumaraswamy distribution based on record values and inter-record times, Journal of Statistical Computation Simulation. 86, 2471-2493 (2016).

[32] S. Dey, T. Dey and D. J. Luckett, Statistical inference for the generalized inverted exponential distribution based on upper record values, Mathematics and Computers in Simulation. 120, 64-78 (2016).

[33] S. Singh, Y. M. Tripathi and S. Wu, Bayesian estimation and prediction based on lognormal record values, Journal of Applied Statistics. 44, 916-940, (2017).

[34] R. S. Ahmed, N. Balakrishnan and J. Ahmadi, Bayesian prediction of order statistics with fixed and random sample sizes based on k-record values from a Pareto distribution. Communications in Statistics-Theory and Methods. 46, 721-735 (2017).

[35] M. Ahsanullah, Record Statistics. Nova Science Plublishers Inc. New York, USA (1995).

Table 10: Two sample prediction for the future order statistics

95% HPD credible intervals for Y_S		
Y_S	[Mean Lower,Mean Upper]	Mean Length
Y_1	$[0.3781, 1.2583]$	0.8802
Y_2	$[0.5249, 1.7257]$	1.2008
Y_3	$[0.6780, 2.0437]$	1.3658
Y_4	$[0.8739, 2.6601]$	1.7862
Y_5	$[1.1624, 3.6736]$	2.5112

Table 5. Average mean and MSEs of different estimates of λ and β.

		λ						β					
				LINEX		GE				LINEX		GE	
(T, n, m)	CS	MLE	SE	$c=-1$	$c=1$	$q=-1$	$q=1$	MLE	SE	$c=-1$	$c=1$	$q=-1$	$q=1$
$(0.8, 50, 40)$	I	3.1136	3.0866	3.0859	2.9707	3.0866	3.0106	3.0773	3.0688	3.0685	3.0122	3.0688	3.0315
		0.3474	0.2699	0.2696	0.2147	0.2699	0.2421	0.1381	0.1161	0.1160	0.1027	0.1161	0.1090
	II	3.1915	3.1614	3.1608	3.0396	3.1614	3.0833	3.1205	3.1065	3.1062	3.0482	3.1065	3.0686
		0.3775	0.2920	0.2915	0.2139	0.2920	0.2506	0.1603	0.1374	0.1373	0.1185	0.1374	0.1271
	III	3.2147	3.1727	3.1720	3.0486	3.1727	3.0937	3.1617	3.1434	3.1431	3.0837	3.1434	3.1051
		0.4423	0.3305	0.3299	0.2423	0.3305	0.2842	0.1835	0.1530	0.1529	0.1287	0.1530	0.1396
$(0.8, 50, 45)$	I	3.1654	3.1352	3.1345	3.0119	3.1352	3.0552	3.1160	3.1026	3.1023	3.0392	3.1026	3.0613
		0.3342	0.2628	0.2624	0.1963	0.2628	0.2271	0.1373	0.1112	0.1111	0.1012	0.1112	0.1003
	II	3.1636	3.1289	3.1283	3.0080	3.1289	3.0507	3.1102	3.0957	3.0954	3.0329	3.0957	3.0548
		0.3515	0.2718	0.2713	0.2110	0.2718	0.2354	0.1400	0.1270	0.1268	0.1080	0.1270	0.1150
	III	3.1531	3.1212	3.1205	3.0023	3.1212	3.0441	3.1067	3.0923	3.0919	3.0299	3.0923	3.0516
		0.3517	0.2774	0.2769	0.2118	0.2774	0.2429	0.1505	0.1262	0.1261	0.1076	0.1262	0.1159
$(0.8, 60, 45)$	I	3.0938	3.0687	3.0681	2.9685	3.0687	3.0028	3.0815	3.0686	3.0684	3.0138	3.0686	3.0326
		0.2741	0.2216	0.2213	0.1701	0.2216	0.2012	0.1318	0.1090	0.1100	0.1007	0.1090	0.1002
	II	3.1634	3.1379	3.1374	3.0351	3.1379	3.0717	3.1068	3.0930	3.0927	3.0374	3.0930	3.0567
		0.2744	0.2231	0.2228	0.1789	0.2231	0.2013	0.1322	0.1114	0.1213	0.1048	0.1114	0.1125
	III	3.1196	3.0928	3.0923	2.993	3.0928	3.0278	3.0838	3.0698	3.0695	3.0143	3.0698	3.0334
		0.2791	0.2253	0.2250	0.1794	0.2253	0.2014	0.1503	0.1258	0.1257	0.1050	0.1258	0.1151
$(1.4, 50, 40)$	I	3.1556	3.1255	3.1248	3.0031	3.1255	3.0460	3.0857	3.0744	3.0742	3.0175	3.0744	3.0370
		0.3912	0.2966	0.2962	0.2222	0.2966	0.2505	0.1382	0.1134	0.1134	0.1000	0.1134	0.1071
	II	3.1645	3.1315	3.1309	3.0123	3.1315	3.0547	3.0788	3.0656	3.0653	3.0086	3.0656	3.0281
		0.3919	0.2971	0.2966	0.2230	0.2971	0.2009	0.1400	0.1148	0.1147	0.1001	0.1148	0.1073
	III	3.3138	3.2673	3.2666	3.1358	3.2673	3.1855	3.0567	3.0411	3.0409	2.9869	3.0411	3.0052
		0.5049	0.3733	0.3726	0.2584	0.3733	0.2116	0.1411	0.1141	0.1140	0.1035	0.1141	0.1089
$(1.4, 50, 45)$	I	3.1454	3.1251	3.1244	2.9977	3.1251	3.0416	3.0627	3.0601	3.0598	2.9988	3.0601	3.0196
		0.3026	0.2394	0.2389	0.1789	0.2394	0.2064	0.1372	0.1111	0.1120	0.1001	0.1111	0.1008
	II	3.1596	3.1255	3.1248	3.0058	3.1255	3.0481	3.0966	3.0808	3.0804	3.018	3.0808	3.0396
		0.3857	0.2374	0.2270	0.1866	0.2374	0.2500	0.1405	0.1132	0.1131	0.1005	0.1132	0.1039
	III	3.1343	3.1050	3.1044	2.9879	3.1050	3.0281	3.0828	3.0695	3.0692	3.0075	3.0695	3.0287
		0.2956	0.2274	0.2270	0.1758	0.2274	0.1998	0.1517	0.1234	0.1233	0.1076	0.1234	0.1150
$(1.4, 65, 45)$	I	3.1632	3.1472	3.1466	3.0314	3.1472	3.0724	3.0850	3.0791	3.0788	3.023	3.0791	3.0424
		0.3017	0.2334	0.2330	0.1988	0.2334	0.2091	0.1374	0.1106	0.1105	0.1008	0.1106	0.1003
	II	3.1557	3.1266	3.1261	3.0241	3.1266	3.0606	3.1121	3.0956	3.0953	3.0388	3.0956	3.0585
		0.3049	0.2490	0.2486	0.1918	0.2490	0.2192	0.1340	0.1129	0.1128	0.0973	0.1129	0.1043
	III	3.0120	3.0021	3.0124	2.9865	3.0021	3.1078	3.0462	3.0456	3.0184	3.0005	3.0456	3.0265
		0.2845	0.2123	0.2245	0.1912	0.2123	0.2198	0.1423	0.1224	0.1213	0.1005	0.1224	0.1120

With each scheme the first value represents the average relative estimate and the second value represents MSE.

Table 6. Average mean and MSEs of different estimates of $S(t)$.

		$S(t)$					
				LINEX		GE	
(T,n,m)	CS	MLE	SE	$c=-1$	$c=1$	$q=-1$	$q=1$
$(0.8,50,40)$	I	0.8811	0.8747	0.8747	0.8741	0.8747	0.8731
		0.0012	0.0011	0.0011	0.0011	0.0011	0.0015
	II	0.8865	0.8800	0.8800	0.8794	0.8800	0.8785
		0.0013	0.0011	0.0011	0.0011	0.0011	0.0012
	III	0.8862	0.8791	0.8791	0.8785	0.8791	0.8776
		0.0014	0.0013	0.0012	0.0012	0.0013	0.0012
$(0.8,50,45)$	I	0.8850	0.8782	0.8782	0.8775	0.8782	0.8766
		0.0012	0.0011	0.0012	0.0012	0.0011	0.0012
	II	0.8841	0.8773	0.8773	0.8767	0.8773	0.8758
		0.0013	0.0013	0.0013	0.0013	0.0013	0.0013
	III	0.8837	0.8770	0.0267	0.8764	0.8770	0.8755
		0.0014	0.0013	0.0013	0.0013	0.0013	0.0013
$(0.8,60,45)$	I	0.8807	0.8749	0.8748	0.8743	0.8749	0.8735
		0.0011	0.0011	0.0011	0.0011	0.0011	0.0011
	II	0.8862	0.8804	0.8804	0.8799	0.8804	0.8792
		0.0011	0.0010	0.0010	0.0010	0.0010	0.0010
	III	0.8832	0.8774	0.8774	0.8768	0.8774	0.8761
		0.0011	0.0010	0.0010	0.0010	0.0010	0.0011
$(1.4,50,40)$	I	0.8838	0.8773	0.8773	0.8766	0.8773	0.8756
		0.0014	0.0012	0.0012	0.0012	0.0012	0.0012
	II	0.8851	0.8785	0.8785	0.8779	0.8785	0.8770
		0.0014	0.0012	0.0012	0.0012	0.0012	0.0012
	III	0.8966	0.8894	0.89	0.8894	0.8894	0.8880
		0.0014	0.0011	0.0011	0.0011	0.0011	0.0011
$(1.4,50,45)$	I	0.8783	0.8783	0.8783	0.8776	0.8783	0.8766
		0.0012	0.0012	0.0012	0.0012	0.0012	0.0012
	II	0.8843	0.8777	0.8777	0.8770	0.8777	0.8762
		0.0012	0.0012	0.0012	0.0012	0.0012	0.0012
	III	0.8840	0.8774	0.8773	0.8767	0.8774	0.8758
		0.0012	0.0011	0.0012	0.0012	0.0011	0.0012
$(1.4,65,45)$	I	0.8857	0.8800	0.8800	0.8793	0.8800	0.8785
		0.0012	0.0012	0.0012	0.0012	0.0012	0.0012
	II	0.8846	0.8790	0.8790	0.8785	0.8790	0.8777
		0.0012	0.0012	0.0012	0.0012	0.0012	0.0012
	III	0.8770	0.8783	0.8845	0.8741	0.8783	0.8643
		0.0012	0.0012	0.0012	0.0012	0.0012	0.0012

With each scheme the first value represents the average relative estimate and the second value represents MSE.

Table 7. Average mean and MSEs of different estimates of $H(t)$.

		$H(t)$					
				LINEX		GE	
(T, n, m)	CS	MLE	SE	$c = -1$	$c = 1$	$q = -1$	$q = 1$
$(0.8, 50, 40)$	I	0.7697	0.7864	0.7864	0.7734	0.7864	0.7523
		0.0295	0.0262	0.0262	0.0255	0.0262	0.0269
	II	0.7585	0.7742	0.7742	0.7614	0.7742	0.7401
		0.0295	0.0267	0.0267	0.0257	0.0267	0.0274
	III	0.7669	0.7852	0.7851	0.7719	0.7852	0.7505
		0.0322	0.0293	0.0293	0.0279	0.0293	0.0293
$(0.8, 50, 45)$	I	0.7638	0.7807	0.7806	0.7672	0.7807	0.7452
		0.0267	0.0244	0.0244	0.0232	0.0244	0.0244
	II	0.7659	0.7829	0.7828	0.7696	0.7829	0.7483
		0.0276	0.0264	0.0264	0.0259	0.0264	0.0251
	III	0.7667	0.7829	0.7828	0.7698	0.7829	0.7486
		0.0296	0.0267	0.0267	0.0254	0.0267	0.0265
$(0.8, 60, 45)$	I	0.7760	0.7895	0.7895	0.7778	0.7895	0.7595
		0.0220	0.0203	0.0203	0.0210	0.0203	0.0204
	II	0.7593	0.7727	0.7726	0.7616	0.7727	0.7435
		0.0223	0.0208	0.0208	0.0220	0.0208	0.0201
	III	0.7655	0.7789	0.7788	0.7679	0.7789	0.7502
		0.0220	0.0205	0.0205	0.0196	0.0205	0.0204
$(1.4, 50, 40)$	I	0.7598	0.7769	0.7768	0.7631	0.7769	0.7404
		0.0315	0.0249	0.0279	0.0278	0.0249	0.0266
	II	0.7523	0.7690	0.7689	0.7564	0.7690	0.7353
		0.0291	0.0256	0.0256	0.0248	0.0256	0.0286
	III	0.6938	0.7138	0.7259	0.7137	0.7138	0.6801
		0.0293	0.0261	0.0236	0.02470	0.0261	0.0289
$(1.4, 50, 45)$	I	0.7719	0.7718	0.7718	0.7570	0.7718	0.7321
		0.0280	0.0240	0.0250	0.0248	0.0240	0.0260
	II	0.7615	0.7776	0.7775	0.7643	0.7776	0.7427
		0.0290	0.0255	0.0255	0.0242	0.0255	0.0275
	III	0.7622	0.7916	0.7780	0.7651	0.7916	0.7441
		0.0267	0.0257	0.0242	0.0230	0.0257	0.0242
$(1.4, 65, 45)$	I	0.7564	0.7709	0.7708	0.7574	0.7709	0.7351
		0.0271	0.0232	0.0220	0.0232	0.0232	0.0251
	II	0.7666	0.7793	0.7792	0.7680	0.7793	0.7497
		0.0270	0.0245	0.0255	0.0232	0.0245	0.0251
	III	0.7615	0.7816	0.7672	0.7741	0.7816	0.7220
		0.0277	0.0247	0.0232	0.0235	0.0247	0.0252

With each scheme the first value represents the average relative estimate and the second value represents MSE.

Table 8. Average mean and MSEs of different estimates of CV.

(T,n,m)	CS			$c=-1$	$c=1$	$q=-1$	$q=1$
$(0.8,50,40)$	I	0.6914	0.6744	0.6740	0.6766	0.6744	0.6736
		0.0242	0.0228	0.0235	0.0257	0.0228	0.0246
	II	0.6708	0.658	0.6577	0.6726	0.658	0.6703
		0.0272	0.0243	0.0240	0.0257	0.0243	0.0249
	III	0.655	0.7100	0.7096	0.6837	0.7100	0.662
		0.0272	0.0265	0.0266	0.0267	0.0265	0.0267
$(0.8,50,45)$	I	0.6720	0.6618	0.6615	0.6745	0.6618	0.6704
		0.0225	0.0228	0.0224	0.0227	0.0228	0.0226
	II	0.6781	0.6696	0.6692	0.6704	0.6696	0.6758
		0.0266	0.0243	0.0232	0.0257	0.0243	0.0246
	III	0.6787	0.6704	0.6699	0.6707	0.6704	0.6164
		0.0262	0.0255	0.0251	0.0252	0.0255	0.0257
$(0.8,60,45)$	I	0.6849	0.6697	0.6694	0.6747	0.6697	0.6724
		0.0220	0.0217	0.0218	0.0219	0.0217	0.0219
	II	0.6753	0.6600	0.6596	0.6760	0.6600	0.6743
		0.0236	0.0227	0.0232	0.0237	0.0227	0.0226
	III	0.6900	0.6743	0.6739	0.6774	0.6743	0.6749
		0.0252	0.0245	0.0241	0.0252	0.0245	0.0247
$(1.4,50,40)$	I	0.6807	0.6660	0.6655	0.6411	0.6660	0.6782
		0.0239	0.0229	0.0235	0.0237	0.0229	0.0237
	II	0.6876	0.6745	0.6740	0.6777	0.6745	0.6745
		0.0268	0.0253	0.0254	0.0257	0.0253	0.0259
	III	0.6968	0.6830	0.6826	0.6762	0.6830	0.6731
		0.0270	0.0255	0.0250	0.0260	0.0255	0.0257
$(1.4,50,45)$	I	0.6951	0.6825	0.6821	0.6722	0.6825	0.6768
		0.0228	0.0219	0.0225	0.0227	0.0219	0.0227
	II	0.6800	0.6735	0.6731	0.6746	0.6735	0.6694
		0.0255	0.0253	0.0254	0.0251	0.0253	0.0251
	III	0.6871	0.6780	0.6775	0.6884	0.6780	0.6832
		0.0253	0.0250	0.0251	0.0252	0.0250	0.0250
$(1.4,65,45)$	I	0.6848	0.6670	0.6667	0.6720	0.6670	0.6798
		0.0223	0.0217	0.0215	0.0217	0.0217	0.0217
	II	0.6720	0.6730	0.6661	0.6714	0.6730	0.6791
		0.0240	0.0221	0.0222	0.0221	0.0221	0.0219
	III	0.6671	0.6740	0.6675	0.6784	0.6740	0.6732
		0.0248	0.0241	0.0251	0.0246	0.0241	0.0244

With each scheme the first value represents the average relative estimate and the second value represents MSE.

Table 9. 95% coverage probabilities for λ , β , $S(t), H(t)$and CV based on different methods.

		λ		β		$S(t)$		$H(t)$		CV	
(T,n,m)	CS	MLE	MCMC	MLE	MCMC	MLE	MCMC	MLE	MCMC	MLE	MCMC
$(0.8,50,40)$	I	0.9600	0.9560	0.9520	0.9500	0.9180	0.9600	0.9400	0.9540	0.9500	0.9560
	II	0.9540	0.9520	0.9440	0.9360	0.9140	0.9440	0.9320	0.9520	0.9440	0.9540
	III	0.9460	0.9460	0.9400	0.9320	0.9080	0.9300	0.9300	0.9340	0.9080	0.9380
$(0.8,50,45)$	I	0.9620	0.9740	0.9560	0.9600	0.9180	0.9620	0.9580	0.9620	0.9520	0.9580
	II	0.9520	0.9540	0.9540	0.9560	0.9000	0.9540	0.9400	0.9560	0.9420	0.9520
	III	0.9460	0.9540	0.9420	0.9540	0.9140	0.9520	0.9320	0.9520	0.9380	0.9500
$(0.8,60,45)$	I	0.9600	0.9620	0.9560	0.9600	0.9520	0.9620	0.9580	0.9600	0.9620	0.9700
	II	0.9520	0.9580	0.9560	0.9560	0.9460	0.9580	0.9440	0.9580	0.9520	0.9540
	III	0.9460	0.9580	0.9520	0.9540	0.9320	0.9540	0.9420	0.9540	0.9380	0.9580
$(1.4,50,40)$	I	0.9580	0.9620	0.9460	0.9540	0.9220	0.9580	0.9500	0.9500	0.9520	0.9560
	II	0.9520	0.9600	0.9400	0.9520	0.9140	0.9520	0.9440	0.9480	0.9500	0.9520
	III	0.9480	0.9580	0.9220	0.9480	0.9140	0.9320	0.942	0.9460	0.9180	0.9380
$(1.4,50,45)$	I	0.9620	0.9680	0.9520	0.9620	0.9400	0.9560	0.9520	0.9620	0.9540	0.9560
	II	0.9560	0.9560	0.9520	0.9600	0.9320	0.9420	0.9420	0.9520	0.9500	0.9540
	III	0.9400	0.9560	0.9480	0.9540	0.9420	0.9460	0.9400	0.9520	0.9480	0.9500
$(1.4,60,45)$	I	0.9740	0.9760	0.9600	0.9640	0.9600	0.9620	0.9620	0.9640	0.9600	0.9740
	II	0.9600	0.9740	0.9540	0.9580	0.9500	0.9580	0.9560	0.9580	0.9580	0.9620
	III	0.9460	0.9540	0.9520	0.9580	0.9500	0.9500	0.9420	0.9460	0.9480	0.9640

Table 11: Two sample prediction for the future lower record values

95% HPD credible intervals for Z_S		
Z_S	[Mean Lower,Mean Upper]	Mean Length
Z_1	$[0.4847, 2.7175]$	2.2328
Z_2	$[0.3714, 2.0253]$	1.6538
Z_3	$[0.3141, 1.2818]$	0.9677
Z_4	$[0.2731, 0.9978]$	0.7247
Z_5	$[0.2535, 0.8206]$	0.5671